MAC-CPTM Situations Project

Situation 42: Sin (2x)

Prepared at Pennsylvania State University
Mid-Atlantic Center for Mathematics Teaching and Learning
18 July 2005 – Heather Godine

Edited at Pennsylvania State University
Mid-Atlantic Center for Mathematics Teaching and Learning
13 February 2007 – Evan McClintock
23 February 2007 – Rose Mary Zbiek

Edited at the University of Georgia
Center for Proficiency in Teaching Mathematics
26 September 2008 – Brian Gleason, Shawn Broderick

Prompt

During a lesson on transformations of the sine function a student asks,
"Why is the graph of $y = \sin(2x)$ a horizontal shrink of the graph of $y = \sin(x)$ instead of a horizontal stretch?"

Commentary

The period of the function $\sin(kx)$, $k \neq 0$ is $\frac{2\pi}{|k|}$. Thus, the case of $k = 2$ represents a horizontal shrink of the function $\sin(x)$, since the period is smaller. The first two foci emphasize the periodicity of the sine function, appealing to composition of functions and to the unit circle. In contrast, the third focus emphasizes the first derivative of $y = \sin(2x)$ as the instantaneous rate of change of the function.

Mathematical Foci

Mathematical Focus 1

The periodic function $g(x) = \sin(2x)$ is a composition of the periodic function $f(x) = \sin(x)$ and the linear function $h(x) = 2x$. The period of g is smaller than the period of f, which is represented graphically by a horizontal 'shrink.'

The composition uses the outputs of $h(x) = 2x$ as inputs of $f(x) = \sin(x)$. Figure 1 provides a table of these inputs and outputs, and it should be apparent from this table that $\sin(2x)$ "progresses" through the cycle of output values twice as fast as $\sin(x)$, because $2x$ "progresses" through its output values twice as fast as x (see the bolded entries). Figure 2 allows us to compare the graphs of these periodic functions, from which it is also apparent that $\sin(x)$ has a larger period, 2π, than does $\sin(2x)$, which has period π. Therefore, the graph of $\sin(2x)$ is a horizontal shrink of the graph of $\sin(x)$.

x	2*x*	sin(*x*)	sin(2*x*)
0	0	0	0
π/12	π/6	0.259	1/2
π/6	π/3	1/2	√3/2
π/4	π/2	√2/2	–1
π/3	2π/3	√3/2	√3/2
5π/12	5π/6	0.966	1/2
π/2	π	1	0
7π/12	7π/6	0.966	–1/2
2π/3	4π/3	√3/2	–√3/2
3π/4	3π/2	√2/2	–1
5π/6	5π/3	1/2	–√3/2
11π/12	11π/6	0.259	–1/2
π	2π	0	0
13π/12	13π/6	–0.259	1/2
7π/6	7π/3	–1/2	√3/2
5π/4	5π/2	–√2/2	1
4π/3	8π/3	–√3/2	√3/2
17π/12	17π/6	–0.966	1/2
3π/2	3π	–1	0
19π/12	19π/6	–0.966	–1/2
5π/3	10π/3	–√3/2	–√3/2
7π/4	7π/2	–√2/2	–1
11π/6	11π/3	–1/2	–√3/2
23π/12	23π/6	-0.259	–1/2
2π	4π	0	0

Figure 1

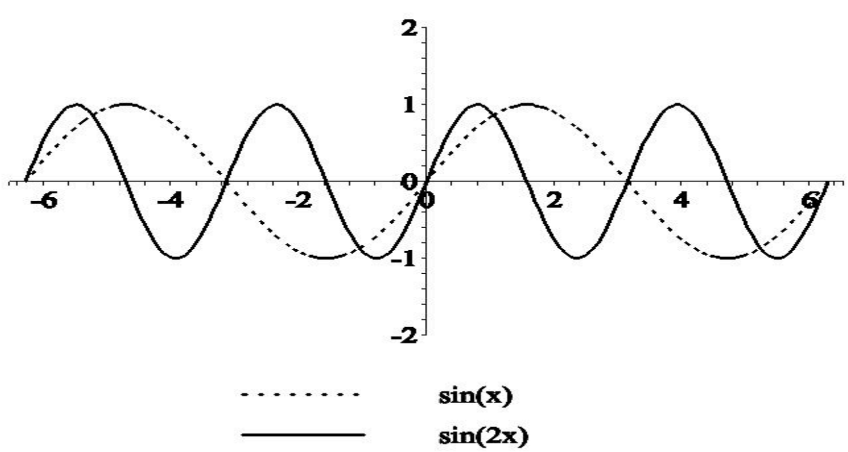

Figure 2

Mathematical Focus 2

The unit circle is the locus of all points with coordinates $(\cos(\theta),\ \sin(\theta))$*, which can allow us to see that the period of* $\sin(\theta)$ *is* 2π *while the period of* $\sin(2\theta)$ *is* π.

Consider the unit circles with embedded triangles ABC and $A'B'C'$ as shown in Figure 3. If θ is allowed to vary from 0 to 2π, then it is apparent that A' will traverse the unit circle twice while A traverses the unit circle only once. Since $|A'C'| = \sin(2\theta)$, this double traversing of the unit circle by A' represents two periods of $\sin(2\theta)$. Similarly, $|AC| = \sin(\theta)$ implies that the single traversing of the unit circle by A represents one period of $\sin(\theta)$. Thus, $\sin(2\theta)$ has half the period of $\sin(\theta)$, and so its graph will be a horizontal shrink (by a factor of $\frac{1}{2}$), of the graph of $\sin(\theta)$

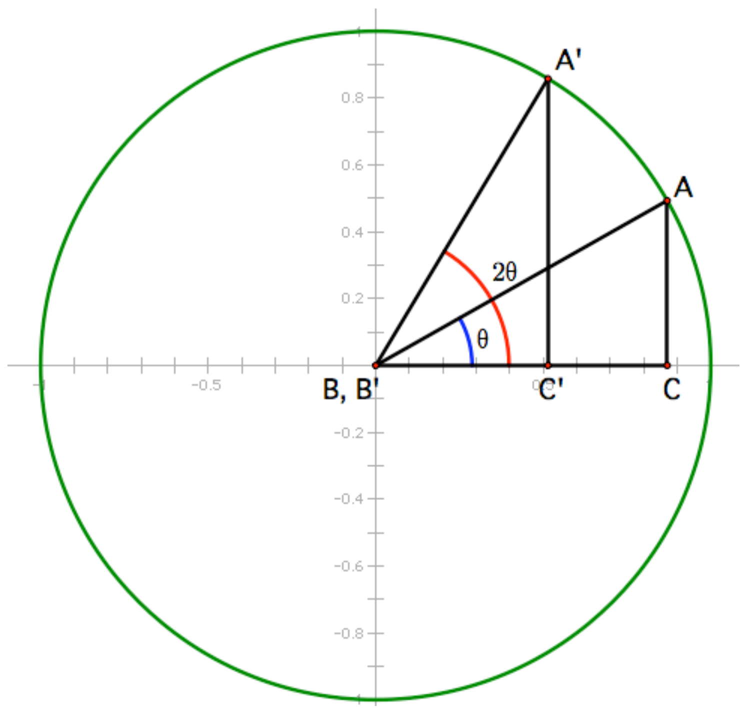

Figure 3

Mathematical Focus 3

The first derivative, as the instantaneous rate of change of a function, and can be used in particular cases to locate maximum and minimum values of a function, and in the case of a regularly oscillating function, determine the period.

Since $\sin(kx)$ is composed of regular oscillations, one may judge its period (and hence its stretch) by how quickly these oscillations occur. A rough measure of this quickness is given by the derivative, since a steeper slope (or greater instantaneous rate of change) implies a faster oscillation. The derivative of $\sin(x)$ is $\cos(x)$, while the derivative of $\sin(2x)$ is $2\cos(2x)$. While these are not immediately comparable because of the factor of 2 on the x, the factor of 2 on the $\cos(\cdot)$ may lead us to believe that $2\cos(2x)$ in general represents a larger value than $\cos(x)$. Unfortunately, a quick look at the graphs of these functions challenges that belief (see figure 4). Nevertheless, we can note that the zeros of $2\cos(2x)$ are closer together than the zeros of $\cos(x)$, and since these zeros represent the relative extrema of the respective $\sin(\cdot)$ functions, we can conclude that the extrema of $\sin(2x)$ are closer together than the extrema of $\sin(x)$, which implies a faster oscillation and hence a horizontal 'shrink.'

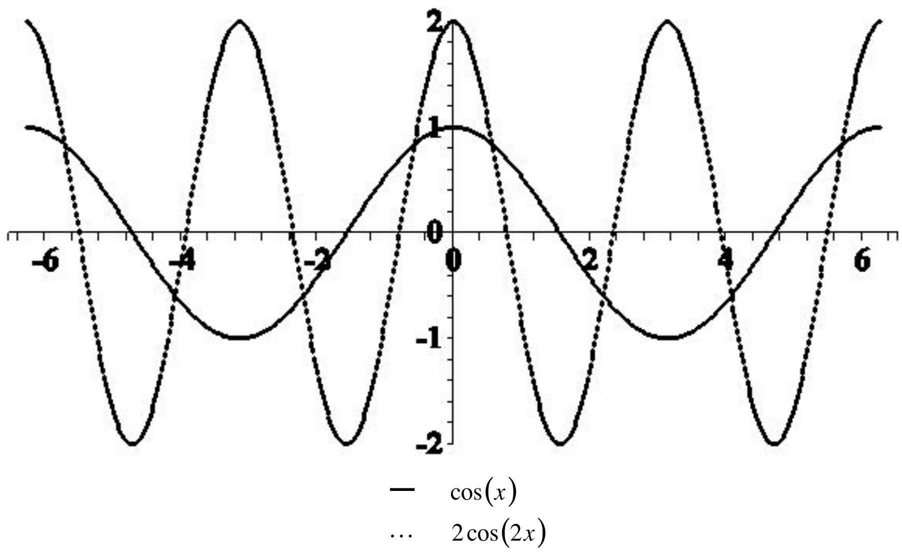

Figure 4

Post Commentary

The set of foci attempt to provide some intuitive understanding for why the graph of $y = \sin(2x)$ is a horizontal shrink of the graph of $y = \sin(x)$ instead of a horizontal stretch. This can be counterintuitive to students because the graph of $y = 2\sin(x)$ actually is a vertical stretch of the graph of $y = \sin(x)$ and because students often associate multiplication with "making things larger," especially when the multiplicative factor is greater than 1.